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Stirling's Approximation

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I. SYNOPSIS

Stirling's approximation to the factorial, which is used in the Lagrange multiplier derivation of the Boltzmann distribution, is explained here.

II. INTRODUCTION

$\ell n\,(N!)$ is the starting point for this derivation, rather than N factorial ($N!$) itself.

$$\ell nN! = \ell nN + \ell n(N-1) + \ell n(N-2)\cdots$$

can be re-written as

$$\ell nN! = \sum_{j=0}^{j=N-1} \ell n(N-j)$$

We can picture this sum as the sum of the lengths of all the vertical lines shown in Figure 1.

We now write this backwards:

$$\ell nN! = \sum_{j=1}^{j=N} \ell n(j)$$

which covers the same territory. This can be seen in Figure 2.

Next, we convert this sum to an area by constucting horizontal bridges (as shown, see Figure 3) where the width of each rectangle is going to turn out to be one (1)! This means that the height and the area are synonymous!

We then have (see Figure 3):

$$\ell nN! = \sum_{j=1}^{j=N} \ell n(j)[(j+1)-j]$$

as an area, and we rewrite this as

$$\ell nN! = \left[sum_{j=1}^{j=N} \ell n(j)\right] \Delta j$$

preparatory to making the histogram to continuous functional area transformation taught in first year calculus. We then have

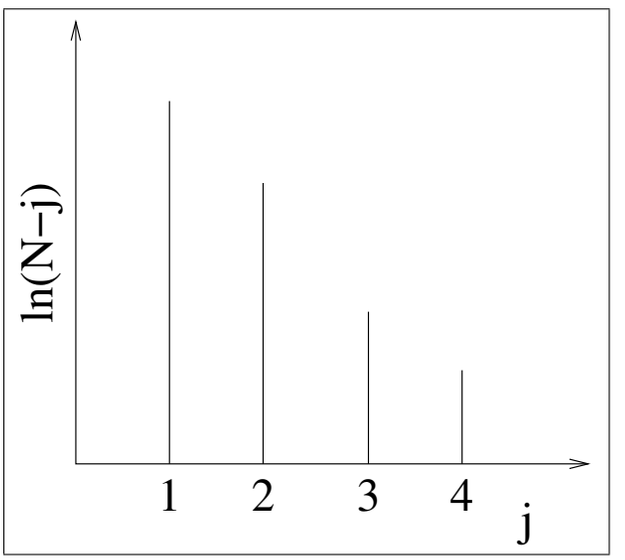

FIG. 1: The logarithmic factorial sum, shown explicitly.

$$\ell nN! = \int_{j=1}^{j=N} \ell n(j)\delta j$$

which is trivially integrable to give

$$\ell nN! = (j\ell nj - j)|_{j=1}^{j=N}$$

which evaluates to

$$\ell nN! = N\ell nN - N - (1\ell n1 - 1)$$

which is, in the limit N much larger than 1

$$\ell nN! = N\ell nN - N$$

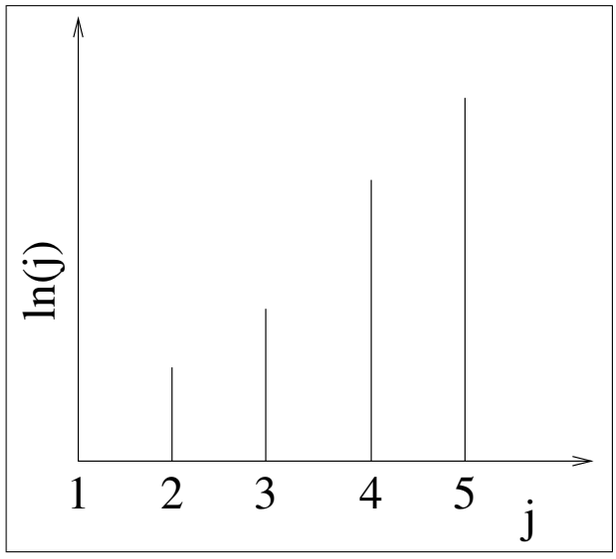

FIG. 2: The logarithmic factorial sum, reversed.

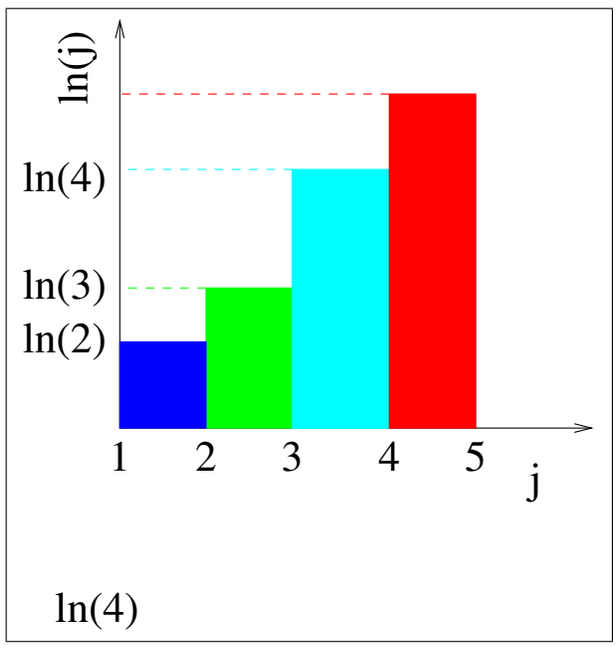

FIG. 3: The logarithmic factorial sum, converted to a histogram.