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A Universal Formula for Continuous Utility

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Abstract

A single formula assigns a continuous utility function to every representable preference relation.

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1. Introduction.

Debreu (1964) characterized preferences $\succsim$ on X representable by a continuous utility function as transitive, complete and strongly separable (there exists a countable, strongly dense subset A of X). Debreu proved the sufficiency of these conditions by constructing a utility function, then showing how to make it continuous in the order topology. Jaffray (1975) shortened the proof by constructing a continuous utility function in two stages. He first constructed u on countable strongly dense $A = \{a_1, a_2, \ldots\}$ inductively. (Number the rationals in $[0,1]$. Let $u(a_i)$ be the first rational in $[0,1]$ with the same relation to $u(a_1), \ldots, u(a_{i-1})$ under $\geq$ that a_i has to $a_1, \ldots, a_{i-1}$ under $\succsim$). Then he defined u on $X - A$ by formula: $u(x) = \sup\{u(a_i)\colon\ x \succsim a_i\}$.

Our goal is not to shorten Jaffray's sufficiency proof, which is already as short as possible, but to produce a single formula, a one-line display that assigns a continuous utility function to every representable preference relation.

2. Preliminaries.

We adopt the version of Debreu's characterization used by Jaffray in which a subset A of X is said to be *strongly dense* if for $x, y \in X$, $x \succ y$ implies $x \succsim a \succ b \succsim y$ for some $a, b \in A$. For any countable $A = \{a_1, a_2, \ldots\} \subseteq X$ define *successor relation* S on X by xSa if $a = a^i$ and either $x \in X - A$ or $x = a^j$ with $j > i$.

For nonempty $T \subseteq \Re$ define $\sup T$ to be the least upper bound of T and $\inf T$ to be the greatest lower bound. Let $\sup \emptyset = 0$ and $\inf \emptyset = 1$.

3. Definition of u.

For transitive, complete $\succsim$ on X, countable, strongly dense $A = \{a_1, a_2, \ldots\} \subseteq X$ and associated successor relation S on X, define $u\colon\ X \to [0,1]$ inductively by

$$u(x) = (\sup\{u(a)\colon\ x \succsim a \text{ and } xSa\} + \inf\{u(a)\colon\ a \succsim x \text{ and } xSa\})/2$$

4. u is a continuous utility function.

The above formula for u implements Jaffray's construction, but with arbitrarily numbered rationals replaced by $\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \ldots$ and also implements his extension formula. Therefore by Jaffray's proof u is a continuous utility function representing $\succsim$ on X.

5. Concluding Remark.

The key that enabled us to combine Jaffray's construction on A and his extension formula into a single formula was the replacement of arbitrarily numbered rationals in his construction by an ordered, countable, dense subset of $[0, 1]$ such that each element after the first is halfway between adjacent (under $\geq$) predecessors or halfway between an outermost predecessor and 0 or 1.

References

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